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Optimal stopping problem with reservation — minimization model —

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Abstract: The author has studied optimal stopping problems where all of offers appearing subsequently and randomly can be reserved as well as accepted or rejected. In these models, he tried to maximize the total discounted net profit obtained during the time horizon. This paper deals with a basic model where minimization of the total cost is the objective, and reveals the properties of an optimal decision rule. A major finding is that almost all the properties in [2] keep also. But the procrastination strategy can become an optimal, though it was not seen in the past maximization models.

1. Introduction

The author has studied optimal stopping problems where offers found are allowed to be reserved as well as accepted and rejected them, but some cost is incured in compensation for an reservation. The past basic model is presented in [2] where every offer reserved allows to be recalled every time you like up to the deadline, that is, there are no expiration dates. Models with expiration dates were analyzed in [3] and [6]. In the former, effective life of each reservation is fixed to k regardless of offer value, and in the latter, it can be selected among $\{1, 2, \ldots, k\}$ but reserving cost depends on the length of reservation as well as offer value.

A common and interesting result of them is that any of offers reserved should not be recalled prior to its maturity of reservation([3][6]) or the deadline of the process([2][3][6]), although any reserved offer is recallable everyday in its available days. Continuing the search increases searching costs and decreases the probability of finding better offers. So, the author feel it is strange result. Because he thought why the strange result causes is the assumption that the cost to recall an offer is not returned even if it is recalled, he reserached in [7] a model without the assumption, that is, the reserving cost returns if the offer is recalled. The result, however, is hold also. We found that both the effective periods of reservation and the restitution of reserving cost are not essential causes of that strange result.

Note to here is that all of them were studied in maximization scheme, that is, selling scheme. So, the author trys to check whether or not the result obtained in the past can be said in minimization scheme, that is, buying scheme.

An interesting result gained in this paper is that, depending on conditions, it may be optimal not to stop the search even when finding an offer with lowest price. Such a property was not found in all of the past models. As for the strange result stated above, it is obtained here also. So even if we change the past models to minimization types, the result will be sure to be found in them. But, not all of result are the same.

The model is precisely defined in section 2. Notation and useful function is introduced in section 3. In section 4 the optimal equation of the model is formulated, and its properties are analysed in section 5. The final section 6 summarizes the properties of the optimal decision rule.

2. Model and Assumptions

You must buy one offer among ones you find within the assigned timetable. Bringing an offer into your sight today requires a *search cost* $s > 0$ paid yesterday. The values of offers appearing subsequently are seem to be i.i.d. random variables following a known distribution F such that $F(w) = 0$ for $w < a$, $0 < F(w) < 1$ for $a \le w < b$, $F(w) = 1$ for $b \le w$ with $0 \le a < b < \infty$, and the expectation is μ.

Every time you find a new offer, called a *current offer*, you will check it and decide how to manage it. The choices available are accepting it, passing it up, and reserving it. Reserving an offer w needs a *reserving cost* $r(w)$, but enables you to recall the offer at any time after the day. Even if an reserved offer is recalled, the cost to reserve it is not returned. Of course, the reserving costs paid for offers not recalled will be never return, and any of the offers passed up is never recalled.

Since only one offer is to be accepted, you should remember the most lucrative offer of ones reserved. Let us call it *leading offer*. Then, the actions which can be taken at each time except for the deadline are summarized as the following four: accepting the current offer and stopping the search (AS), reserving the current offer and continuing the search (RC), passing up the current offer and stopping the search by recalling the leading offer (PS), and passing up the current offer and continuing the search (PC), where AS, RC, PS, and PC represent the four decisions, respectively. At the deadline, only decisions AS and PS are permitted.

In the model, we consider the value of time by a discount factor β such as $0 < \beta \leq 1$, that is, the present value of q monetary units obtained at the next time is given by βq monetary units.

The objective here is to find an optimal decision rule that guides you to which action should be taken at each decision point so as to minimize the total expected discounted present cost to be paid in the process ahead. That cost is the expectation of the present discounted cost for the offer you accept or recall, plus the expectation of the amount of search costs and reserving costs paid over the periods from the present point in time to the termination of the search.

Finally, let us introduce the following two assumption.

(1) $\beta\mu + s < b$.
(2) $r(w)$ is continuous, nondecreasing and concave in w, and $0 < r(w) < w$ for all $w > 0$.
(3) $r(w) = 0$ for $w \leq 0$ and $r(w) = r(b)$ for $b \leq w$.

The first assumption: Since $\beta\mu + s$ is the expected discounted cost from one more search, nobody may want to keep searching under $b \leq \beta\mu + s$. We can prove theoretically that if $b \leq \beta\mu + s$, then "stopping the search just now" is the optimal decision at everytime, but the proof is omitted.

The second assumption: Continuation is only for technical reasons. Nondecreasingness and $0 < r(w) < w$ are intuitively clear. Concavity is explained as follows. The most likely reserving cost $r(w)$ is a certain fixed rate, say 5%, of w. But let us allow to reserve an offer by smaller rate of its cost if it is expensive, that is, undesirable. Contrary, a large rate reserving cost may be required for an inexpensive offer.

The third assumption: Only for for analytic reasons.

3. Preliminaries

For convenience, we introduce the following two numbers.

$$\alpha = \beta\mu + s \qquad (< b \quad \text{by assumption (1)}) \tag{3.1}$$

$$\rho = \frac{s}{1-\beta} - a \qquad (\beta < 1) \tag{3.2}$$

In the case of $\beta = 1$, let $\rho = \infty$. Furthermore, define a function B as

$$B(x) = \int_a^b \min\{w, x\} \mathrm{d}F(w) \tag{3.3}$$

and a number θ as the solution of the equation $\beta B(x) - x + s = 0$, if it exists. Thus, if θ exists, we have

$$\beta B(\theta) - \theta + s = 0.$$

Lemma 3.1 (a) $B(x)$ is continuous, concave and nondecreasing in x. Especially, $B(x)$ is strictly increasing in $x \leq b$.

(b) $B(x) = x$ for $x \leq a$, $B(x) < x$ for $a < x$, and $B(x) = \mu$ for $b \leq x$.

(c) $B(x) - x$ is nonincreasing in x, and strictly decreasing in $x \geq a$.

(d) θ exists uniquely with $0 < \theta \leq \alpha$. Especially, if $\rho > 0$, then $a < \theta \leq \alpha$.

PROOF. See Ikuta [1]. ∎

4. Optimal Equation

Let point in time t, simply referred to as *time* t later on, be equally spaced and numbered backward from the deadline $t = 0$, thus t also represents the number of periods remaining.

Suppose that we are at time t with having an offer x as the leading offer and w as the current offer. Then, if taking decision AS, we stop searching by buying the offer w. If taking decision PS, we terminate searching by recalling the offer x. When we choose decision PC, we are to continue the search by paying the search cost s, and the leading offer of the next point in time remains to be x.

As for decision RC, we first pay the reserving cost $r(w)$ for the offer w, and then spend the search cost s. The leading offer of the next time is the larger one between w and x, that is, $\max\{w, x\}$. But it can be easily shown that at any time, we have no need to reserve an offer as long as its value is inferior to that of the leading offer of that time. Hence, if taking decision RC at time t, we can regard w as the leading offer of next time, that is, time $t - 1$.

Let us denote $u_t(w,x)$ the minimum total expected present discounted cost by starting time t on which we have a current offer w and the leading offer x. Then, it follows that

$$u_t(w,x) = \min \left\{ \begin{array}{lll} \text{AS} & : & w \\ \text{RC} & : & \beta v_{t-1}(w) + r(w) + s \\ \text{PS} & : & x \\ \text{PC} & : & \beta v_{t-1}(x) + s \end{array} \right\}, \qquad t \geq 1 \tag{4.1}$$

where

$$v_t(x) = \int_a^b u_t(w,x)\mathrm{d}F(w), \quad t \geq 0; v_{-1}(x) = -\infty$$

Obviously, $u_0(w,x) = \min\{w,x\}$, and thus $v_0(x) = B(x)$.

Let us define two functions $z_t^o(x)$ and $z_t^r(w)$ as follows.

$$z_t^o(x) = \min\{x,\ \beta v_{t-1}(x) + s\}, \qquad t \geq 1 \tag{4.2}$$

$$z_t^r(w) = \min\{w,\ \beta v_{t-1}(w) + r(w) + s\}, \qquad t \geq 1 \tag{4.3}$$

Clearly, $z_0^o(x) = x$ and $z_0^r(w) = w$. Due to (4.1), the meaning of $z_t^o(x)$ can be said that the total expected present discounted net cost to be paid by passing up the current offer w at time t, and then following the optimal strategy. Similar, but inverse explanation is taken for $z_t^r(w)$. Therefore, the set of current offers to be accepted or reserved can be denoted by

$$W_t(x) = \{w : z_t^r(w) \leq z_t^o(x)\}, \qquad t \geq 0 \tag{4.4}$$

From above, $u_t(w,x) = \min\{z_t^r(w), z_t^o(x)\}$ for $t \geq 0$, and

$$v_t(x) = \int_a^b \min\{z_t^r(w),\ z_t^o(x)\}\mathrm{d}F(w) = \int_{W_t(x)} z_t^r(w)\mathrm{d}F(w) + \int_{W_t(x)^c} z_t^o(x)\mathrm{d}F(w), \qquad t \geq 0$$

Finally, we introduce functions g_t and f_t as follows.

$$g_t(x) = \beta v_{t-1}(x) - x + s, \qquad t \geq 1 \tag{4.5}$$

$$f_t(x) = \beta v_{t-1}(x) - x + s + r(x), \qquad t \geq 1 \tag{4.6}$$

And let θ_t and λ_t with $t \geq 1$ be the respective roots of $g_t(x) = 0$ and $f_t(x) = 0$, if they exist, that is

$$g_t(\theta_t) = \beta v_{t-1}(\theta_t) - \theta_t + s = 0 \qquad\qquad f_t(\lambda_t) = \beta v_{t-1}(\lambda_t) - x + s + r(\lambda_t)$$

From (4.2) we find that θ_t is a point of indifference between accepting the leading offer x and continuing the search. From (4.3) we know that λ_t is a point of indifference between accepting the current offer w and reserving it.

Now, note that the optimal decision rule can be described by using $W_t(x)$, θ_t and λ_t.

5. Analysis

Lemma 5.1 (a) $v_t(x)$ is continuous, nondecreasing and concave in $x \geq 0$; and nonincreasing in $t \geq 0$.
(b) $v_t(x) < x$ for $a < x$ and $v_t(x) \leq \mu$ for any x; and $v_t(x)$ is constant while $x \geq b$.

PROOF. (a) It follows from $v_0(x) = B(x)$ and Lemma 3.1(a) that the assertion holds true for $t = 0$.

So as to proceed an induction on t, suppose that $v_{t-1}(x)$ has the three properties. For any fixed w, by noting that $\min\{w,x\}$ has the same properties, we find all the four terms in braces of $u_{t-1}(w,x)$, thus $u_{t-1}(w,x)$ itself also has all the properties. Hence, so does $v_t(x)$.

As for the latter part, we can easily confirm the assertion by induction on t starting with the fact that $u_0(w,x) \geq u_1(w,x)$ for any w and x by definition of u.

(b) Assertion (a) and Lemma 3.1(b) shows $x > B(x) = v_0(x) \geq v_1(x) \geq v_2(x) \geq \cdots$ for any $x > a$. For the latter part, since $u_t(w) \leq w$ for any w and x, we get $v_t(x) = \int_a^b u_t(w,x)\mathrm{d}F(w) \leq \int_a^b w\mathrm{d}F(w) = \mu$.

As for the latter part, it is clear for $t = 0$ due to $v_0(x) = B(x)$ and 3.1(b).

Let $x \geq b$ and suppose $v_t(x) = c_{t-1}$. Then, for any $w \in [a,b]$ we have $w \leq x$, thus $u_t(w,x) = \min\{w, \beta v_{t-1}(w) + r(w) + s, \beta c_{t-1} + s\}$, which is independent of x. Thus, $v_t(x)$ is also independent of x. ∎

Lemma 5.2 Suppose $\rho \le 0$. Then

(a)

$$v_t(x) = \beta^t\Big(x - \frac{s}{1-\beta}\Big) + \frac{s}{1-\beta} \qquad \text{for } x \le a \tag{5.1}$$

(b) $v_t(x) - x$ is strictly decreasing in x.

PROOF. Note that the assumption $\rho \le 0$ holds only when $\beta < 1$, thus $\beta^t - 1 < 0$ for all $t \ge 1$.

(a) Since $v_0(x) = B(x)$, it follows from Lemma 3.1(b) that $v_0(x) = \beta^0(x - s/(1-\beta)) + s/(1-\beta)$ for $x \le a$.

Suppose $v_{t-1}(x) = \beta^{t-1}(x - s/(1-\beta)) + s/(1-\beta)$ for $x \le a$. Then, for $x \le a$,

$$\beta v_{t-1}(x) + s = \beta\Big(\beta^{t-1}\Big(x - \frac{s}{1-\beta}\Big) + \frac{s}{1-\beta}\Big) + s = \beta^t\Big(x - \frac{s}{1-\beta}\Big) + \frac{s}{1-\beta} \tag{5.2}$$

Furthermore, for $x \le a$,

$$\beta^t\Big(x - \frac{s}{1-\beta}\Big) + \frac{s}{1-\beta} - x = (\beta^t - 1)\Big(x - \frac{s}{1-\beta}\Big) < (\beta^t - 1)\Big(a - \frac{s}{1-\beta}\Big) < 0 \tag{5.3}$$

From (5.2) and (5.3) we arrive at

$$\beta^t\Big(x - \frac{s}{1-\beta}\Big) + \frac{s}{1-\beta} = \beta v_{t-1}(x) + s < x \tag{5.4}$$

If $x \le a$, we have $x \le w$ for any $w \in [a, b]$. Due to Lemma 5.1(a) and assumption $r(w) > 0$, we have $\beta v_{t-1}(x) + s < \beta v_{t-1}(w) + r(w) + s$ for any $w \in [a, b]$. By combining these two facts and (5.4), we conclude

$$u_t(w, x) = \beta v_{t-1}(x) + s = \beta^t\Big(x - \frac{s}{1-\beta}\Big) + \frac{s}{1-\beta}$$

which immediately implies (5.1). We have thus completed the induction.

(b) It follows from (a) and (5.3) that, for $x \le a$,

$$v_t(x) - x = \beta^t\Big(x - \frac{s}{1-\beta}\Big) + \frac{s}{1-\beta} - x = (\beta^t - 1)\Big(x - \frac{s}{1-\beta}\Big) < 0$$

that is, $v_t(x) < x$. By adding this fact and Lemma 5.1(b) we find $v_t(x) < x$ for all x. Furthermore, since $\beta^t - 1 < 0$, we know that $v_t(x) - x$ is strictly decreasing in $x \le a$.

Choose x^1 and x^2 so that $0 \le x^1 < x^2$. The concavity of $v_t(x)$, provided at Lemma 5.1(a), yields

$$\frac{v_t(x^2) - v_t(x^1)}{x^2 - x^1} \le \frac{v_t(x^2) - v_t(0)}{x^2 - 0} \tag{5.5}$$

Substitute the facts $v_t(x^2) < x^2$ and

$$v_t(0) = \beta^t\Big(0 - \frac{s}{1-\beta}\Big) + \frac{s}{1-\beta} = (1 - \beta^t)\frac{s}{1-\beta} > 0$$

into (5.5) we get

$$\frac{v_t(x^2) - v_t(x^1)}{x^2 - x^1} \le \frac{v_t(x^2) - v_t(0)}{x^2 - 0} < \frac{x^2 - v_t(0)}{x^2 - 0} < \frac{x^2 - 0}{x^2 - 0} = 1$$

Since it indicates $v_t(x^2) - v_t(x^1) < x^2 - x^1$, or $v_t(x^1) - x^1 > v_t(x^2) - x^2$, we conclude that $v_t(x) - x$ is strictly decreasing in $x \ge 0$. Since this property for $x \le a$ has already been shown, the proof is finished. ∎

Lemma 5.3 Suppose $\rho > 0$. Then

(a) $v_t(x) = x$ for $x \le a$.

(b) $v_t(x) - x$ is equal to 0 while $x \le a$ and is strictly decreasing while $a \le x$.

PROOF. (a) We can prove it by imitating the induction used in the proof of Lemma 5.2(a).

(b) The former part is clear due to (a). The latter part can be shown in a similar fashion to the proof of Lemma 5.2(b). That is, applying the concavity of $v_t(x)$, but using $v_t(a) = a$ instead of $v_t(0)$. ∎

Corollary 5.4 (a) $z_t^o(x)$ is continuous, nondecreasing and concave in x; and nonincreasing in t.
(b) $z_t^r(x)$ is continuous and nondecreasing in x; and nonincreasing in t.

PROOF. (a,b) Evident due to Lemma 5.1(a). ∎

Corollary 5.5 For $t \geq 1$,
(a) $g_t(x)$ is continuous and concave in x, and strictly decreasing in $x \geq a$. Especially, if $\rho \leq 0$, it is strictly decreasing everywhere.
(b) $f_t(x)$ is continuous and concave in x; and nonincreasing in t.

PROOF. (a) Continuity and concavity are obvious due to Lemma 5.1(a). The other properties are derived from Lemmas 5.2(b) and 5.3(b).

(b) They are evident from Lemma 5.1(a) and the premise of the reserving cost function r. ∎

Lemma 5.6 (a) θ_t exists uniquely for all $t \geq 1$. And if $\rho \leq 0$, then $0 < \theta_t \leq a$, or else $a < \theta_t \leq \alpha$.
(b) λ_t exists uniquely for all $t \geq 1$ with $\theta_t < \lambda_t$ and nonincreasing in t. And if $\rho \leq 0$ and $r(a) \leq (\beta^t - 1)\rho$, then $\theta_t < \lambda_t \leq a$, or else $a < \lambda_t$.

PROOF. (a) Let $\rho \leq 0$. Due to Lemmas 5.2(a) and 5.3(a) we get $g_t(0) = (1 + \beta + \beta^2 + \cdots + \beta^{n-1})s > 0$ and $g_t(a) = (1 - \beta^t)\rho \leq 0$. These and Corollary 5.5(a) gives $0 < \theta_t \leq a$.

In the case of $\rho > 0$, we similarly get $g_t(a) = s - (1 - \beta)a > 0$. Furthermore, Lemmas 5.1(b) gives $g_t(\alpha) = \beta v_{t-1}(\alpha) - (\beta\mu + s) - s = \beta(v_{t-1}(\alpha) - \mu) \leq 0$. Hence, $a < \theta \leq \alpha$.

(b) By definition, we have $f_t(\theta_t) = g_t(\theta_t) + r(\theta_t) = 0 + r(\theta_t) > 0$. From Lemma 5.1(b) we get $f_t(x) \leq \beta\mu - x + s + b$, thus $\lim_{x\to\infty} f_t(x) < 0$ as $x \to \infty$. By adding these and Corollary 5.5(b), we conclude that λ_t exists with $\lambda_t > \theta_t$.

Moreover, since $f_t(x)$ is concave in x, we find that $f_t(x)$ is decreasing in x in neighborhood of λ_t. From this and Corollary 5.5(b) enables us to say that λ_t is nonincreasing in t.

Suppose $\rho > 0$. Then, $(\beta^t - 1)\rho \leq 0 < r(a)$ always holds. Furthermore, $a < \theta_t$ holds by (a), thus $a < \lambda_t$. Contrary, suppose $\rho \leq 0$. Since it follows from (a) that $f_t(a) = g_t(a) + r(a) = (1 - \beta^t)\rho + r(a)$, if $r(a) \leq (\beta^t - 1)\rho$, then $f_t(a) \leq 0$, or else $f_t(a) > 0$. Hence, $\lambda_t \leq a$ holds if and only if $\rho \leq 0$ and $r(a) \leq (\beta^t - 1)\rho$. ∎

Due to Corollary 5.5 and the fact that $f_t(x)$ is strictly decreasing near λ_t, shown in the proof of Lemma 5.6(b), we conclude that if $x \leq \theta_t$, then $z_t^o(x) = x$, or else $z_t^o(x) = \beta v_{t-1}(x) + s$, and that if $w \leq \lambda_t$, then $z_t^r(w) = w$, or else $z_t^r(w) = \beta v_{t-1}(w) + s + r(w)$.

Now, we can prescribe the optimal decision rule as follows.

Optimal decision rule: Suppose that you are at time t on which you already have the leading offer x and draw an offer w. Then, the optimal choices are:

(a) if $w \in W_t(x)$, then:
- if $w \leq \lambda_t$, AS (accept current offer w and stop the search)
- if $\lambda_t < w$, RC (reserve current offer w and continue the search)

(b) if $w \notin W_t(x)$, then:
- if $x \leq \theta_t$, PS (pass up current offer w and stop the search by accepting the leading offer x)
- if $\theta_t < x$, PC (pass up current offer w and continue the search)

The left of Figure 5.1 illustrates an image of the rule. The right is the one for maximization models.

Note that the case, for example, $\lambda_t < a$ is possible to happen. In this case, you will prefer to take a reservation rather than an acceptance for any current offer. This and similar relations are stated later.

Let us continue the analysis.

Lemma 5.7 For $t \geq 1$, $z_t^r(w) = w$ for $w \leq \theta_t$ and $z_t^r(w) > \theta_t$ for $\theta_t < w$.

PROOF. If $w \leq \theta_t$, then $w \leq \lambda_t$, thus $z_t^r(w) = \min\{w, \beta v_{t-1}(w) + r(w) + s\} = w$.

If $\theta_t < w$, it follows from Lemma 5.1(a) that $\beta v_{t-1}(w) + r(w) + s \geq \beta v_{t-1}(\theta_t) + r(w) + s > \beta v_{t-1}(\theta_t) + s = \theta_t$ thus $z_t^r(w) = \min\{w, \beta v_{t-1}(w) + r(w) + s\} > \min\{\theta_t, \theta_t\} = \theta_t$. ∎

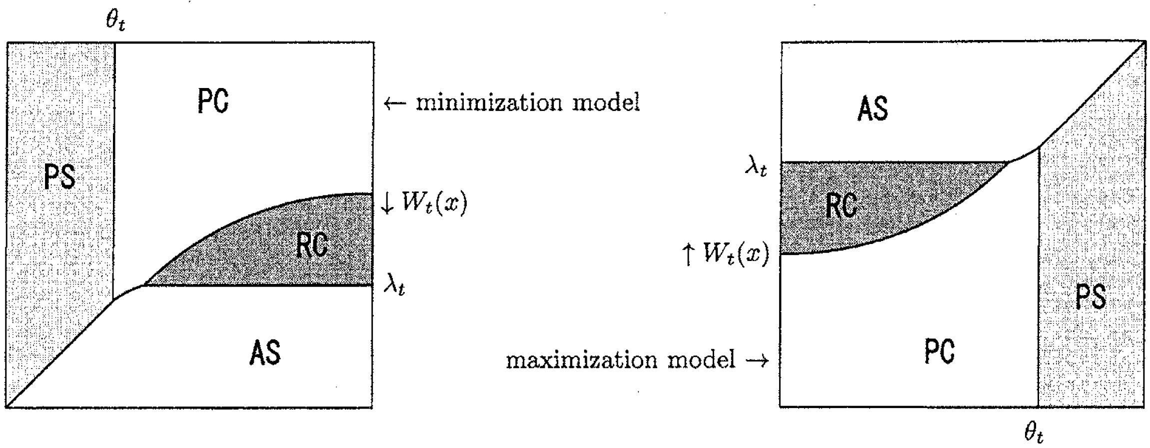

Figure 5.1: Optimal decision rule

Lemma 5.8 $\theta_t = \theta_{t+1}$ if and only if $v_{t-1}(\theta_t) = v_t(\theta_t)$.

PROOF. If $\theta_t = \theta_{t+1}$, we get $g_t(\theta_t) = 0$ and $g_{t+1}(\theta_t) = g_{t+1}(\theta_{t+1}) = 0$. Hence, $g_t(\theta_t) = g_{t+1}(\theta_t)$, or equivalently, $\beta v_{t-1}(\theta_t) + s - \theta_t = \beta v_t(\theta_t) + s - \theta_t$, from which $v_{t-1}(\theta_t) = v_t(\theta_t)$.

Conversely, if $v_{t-1}(\theta_t) = v_t(\theta_t)$, we have $g_{t+1}(\theta_t) = \beta v_t(\theta_t) + s - \theta_t = \beta v_{t-1}(\theta_t) + s - \theta_t = g_t(\theta_t) = 0$. Since θ_{t+1} is the unique solution of $g_{t+1}(x) = 0$, we deduce $\theta_t = \theta_{t+1}$. ∎

Theorem 5.9 $\theta_t = \theta$ for all $t \geq 1$.

PROOF. Throughout the proof, we assume that $\theta_t \in (a, b)$. By definition, $v_0(\theta_1) = \int_a^b \min\{w, \theta_1\} \mathrm{d}F(w) = \int_a^{\theta_1} w \mathrm{d}F(w) + \int_{\theta_1}^b \theta_1 \mathrm{d}F(w)$. By definition of θ_1, we have $z_1^o(\theta_1) = \theta_1$. From this and Lemma 5.7 we get

$$\begin{aligned} v_1(\theta_1) = \int_a^b \min\{z_1^r(w), \theta_1\}\mathrm{d}F(w) &= \int_a^{\theta_1} \min\{z_1^r(w), \theta_1\}\mathrm{d}F(w) + \int_{\theta_1}^b \min\{z_1^r(w), \theta_1\}\mathrm{d}F(w) \\ &= \int_a^{\theta_1} \min\{w, \theta_1\}\mathrm{d}F(w) + \int_{\theta_1}^b \theta_1 \mathrm{d}F(w) \\ &= \int_a^{\theta_1} w\mathrm{d}F(w) + \int_{\theta_1}^b \theta_1 \mathrm{d}F(w) \end{aligned}$$

Hence, $v_0(\theta_1) = v_1(\theta_1)$, thus $\theta_1 = \theta_2$ due to Lemma 5.8.

Suppose $\theta_t = \theta_{t+1}$, or $v_{t-1}(\theta_t) = v_t(\theta_t)$. Then

$$\begin{aligned} v_t(\theta_{t+1}) = v_t(\theta_t) &= \int_a^b \min\{z_t^r(w), \theta_t\}\mathrm{d}F(w) \\ &= \int_a^{\theta_t} w\mathrm{d}F(w) + \int_{\theta_t}^b \theta_t \mathrm{d}F(w) \\ &= \int_a^{\theta_{t+1}} w\mathrm{d}F(w) + \int_{\theta_{t+1}}^b \theta_{t+1}\mathrm{d}F(w) \\ &= \int_a^b \min\{z_{t+1}^r(w), \theta_{t+1}\}\mathrm{d}F(w) = v_{t+1}(\theta_{t+1}) \end{aligned}$$

from which $\theta_{t+1} = \theta_{t+2}$. We have thus confirmed that $\theta_t = \theta_{t+1}$ for all $t \geq 1$.

Let us consider θ_1. Since $g_1(x) = L(x)$, we conclude $\theta_1 = \theta$. Therefore, $\theta_t = \theta$ for all $t \geq 1$. ∎

From now on, we use θ instead of θ_t.

Theorem 5.10 (a) For all t, $W_t(x) \subseteq \{w : w \leq x\}$.
(b) If $x^1 < x^2$, then $W_t(x^1) \subseteq W_t(x^2)$.

PROOF. (a) It is clear for $t=0$ by definitions of $W_0(x)$, $z_t^o(x)$ and $z_t^r(w)$. The case for $t \geq 1$ shall be proven by contraposition.

If $x < w$, then $v_{t-1}(x) \leq v_{t-1}(w)$ due to Lemma 5.1(a), thus $\beta v_{t-1}(x) + s \leq \beta v_{t-1}(w) + s < \beta v_{t-1}(w) + s + r(w)$. Hence, $z_t^o(x) = \min\{x, \beta v_{t-1}(x) + s\} < \min\{w, \beta v_{t-1}(w) + s + r(w)\} = z_t^r(w)$, which indicates $w \notin W_t(x)$. Thereby, the assertion proves to be true.

(b) Let $x^1 < x^2$. We shall show that any $w \in W_t(x^1)$ is also an element of $W_t(x^2)$. Choose any $w \in W_t(x^1)$. Then, due to (4.4) we have $z_t^r(w) \leq z_t^o(x^1)$. From this and Corollary 5.4(a) we get $z_t^r(w) \leq z_t^o(x^1) \leq z_t^o(x^2)$. This means that $w \in W_t(x^2)$. ∎

Theorem 5.11 For $x \leq \theta$

(a) $W_t(x) = \{w : w \leq x\}$.

(b) $u_t(w, x) = \min\{w, x\}$.

(c) $v_t(x) = B(x)$.

PROOF. Note that $\theta_t = \theta$ by Theorem 5.9. So, if we assume $x \leq \theta$, it follows that

$$z_t^o(x) = \min\{x, \beta v_{t-1}(x) + s\} = x \leq \theta$$

(a) We already have $W_t(x) \subseteq \{w : w \leq x\}$ in Theorem 5.10. So, what to be shown here is $\{w : w \leq x\} \subseteq W_t(x)$ for $x \leq \theta$. Choose any w with $w \leq x$. Then, $w \leq x \leq \theta < \lambda_t$ holds, thus $z_t^r(w) = w \leq x = z_t^r(x)$. Hence, $\{w : w \leq x\} \subseteq W_t(x)$.

(b) If $x \leq w \leq \lambda_t$, then $z_t^o(x) = x \leq w = z_t^r(w)$. And if $\lambda \leq w$, then $z_t^o(x) = x \leq \lambda_t = z_t^r(\lambda_t) \leq z_t^r(w)$ due to Corollary 5.4(b). Therefore, while $x \leq w$, we have $z_t^o(x) = x \leq z_t^r(w)$.

Conversely, if $w \leq x$, then since $w \leq x \leq \theta < \lambda_t$ holds, we get $z_t^r(w) = w \leq x = z_t^o(x)$.

Consequently, we obtain $z_t^o(x) = x \leq z_t^r(w)$ for $x \leq w$, and $z_t^r(w) = w \leq x = z_t^o(x)$ for $w \leq x$. Hence, $u_t(w, x) = \min\{z_t^r(w), z_t^o(x)\} = \min\{w, x\}$.

(c) The assertion is immediately derived from (b) and (3.3). ∎

Lemma 5.12 If $r(b) \leq b - \alpha$, then $\lambda_t \leq b$, and if $b - \theta \leq r(b)$, then $b \leq \lambda_t$.

PROOF. By Lemma 5.1(b) we get $f_t(b) = \beta v_{t-1}(b) - b + s + r(b) \leq \beta\mu + s - b + r(b) = \alpha - b + r(b)$. Due to assumption $\alpha < b$, Lemma 3.1(d) and Theorem 5.11(c) we obtain $v_{t-1}(b) \geq v_{t-1}(\theta) = B(\theta)$. Therefore, $f_t(b) \geq \beta B(\theta) + s - b + r(b) = \theta - b + r(b)$. Note that $f_t(x)$ is decreasing near λ_t as seen in the proof of Lemma 5.6(b). Hence, if $r(b) \leq b - \alpha$, then $f_t(b) \leq 0$, thus $\lambda_t \leq b$. And if $b - \theta \leq r(b)$, then $0 \leq f_t(b)$, thus $b \leq \lambda_t$. ∎

Corollary 5.13 Depending on conditions r, the following relations hold.

	$0 < \rho$	$\rho \leq 0$	
		$(\beta^t - 1)\rho < r(a)$	$r(a) \leq (\beta^t - 1)\rho$
$r(b) \leq b - \alpha$	$a < \theta < \cdots \leq \lambda_2 \leq \lambda_1 \leq b$	$\theta \leq a < \cdots \leq \lambda_2 \leq \lambda_1 \leq b$	$\theta < \cdots \leq \lambda_2 \leq \lambda_1 \leq a$
$b - \theta \leq r(b)$	$a < \theta < b \leq \cdots \leq \lambda_2 \leq \lambda_1$	$\theta \leq a < b \leq \cdots \leq \lambda_2 \leq \lambda_1$	impossible

PROOF. The table is derived from Lemmas 5.6, 5.12 and Theorem 5.9. ∎

Theorem 5.14 As $t \to \infty$, we have $v_t(x) \to \min\{B(x), B(\theta)\}$.

PROOF. Omitted. ∎

6. Conclusions

Many of results obtained in the past maximization models([2][3][4][5][6][7]) can be stated in this minimization model. The only result which cannot be stated in the past models is the following.

1. There exist cases where Decisions AS and/or PS are not optimal at all.

In maximization models, no matter what leading offer we have, both Decisions AS and PS can be optimal for certain current offers. Although decision PS is not optimal virtually (see result 2), if we illustrate the optimal decision rules, both decisions always occupy certain areas, respectively.

The result above says that both AS and PS, or only PS is not appear in the figure of the optimal decision rules. From Corollary 5.13 we summarize the decisions possible to take as follows. See Figure 5.1 also.

	$0<\rho$	$\rho \leq 0$	
		$(\beta^t-1)\rho < r(a)$	$r(a) \leq (\beta^t-1)\rho$
$r(b) \leq b-\alpha$	AS,RC,PS,PC	AS,RC,PC (no PS)	RC,PC (no AS,PS)
$b-\theta \leq r(b)$	AS,PS,PC (no RC)	AS,PC (no RC,PS)	impossible

Note that $\rho \leq 0$ holds only when $\beta < 1$ and $s > 0$ seems to be small compared with a, that the decision of termination immediately incurs the expenditure, that is, lose chances to operate the funds. Hence, there may exist the case where even if finding an offer with the lowest value a at intermediate times, we pass it because we want to procrastinate the expenditure.

The following three results had be stated also in the maximization models.

2. An offer reserved during the search process must not be accepted prior to the deadline.

This can be obtained from Theorem 5.9 and Lemma 5.6(b), but we omit the idea to get it. See [2, p.98].

Although reserved offers facilitate us to give up the search at any time, they work only as assurances to hedge the risk that a very expensive offer appears at the deadline. This is the most interesting result in the models with maximization and minimization.

Since the possibility to find the lowest offer decreases as time elapses, and search costs are required everyday up to the termination of the search, the author always think it likely happens that "it seems waste of time for pursuing the search further, I now decide to stop the search by recalling the offer reserved before" comes at a time prior to the deadline. But, the result above holds in this model also.

3. An offer to be either accepted or reserved must have a value superior to the value you can obtain at the time of recalling the leading offer.

This result is the explanation of Theorem 5.10(a) where "superior to" means "lower than or equal to" in this minimization model.

4. After you renew the leading offer, you must choose an offer to be either accepted or reserved more severely.

This result is derived from 5.10(b).

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